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Series expansions of the percolation probability for directed square and honeycomb lattices

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Abstract. We have derived long series expansions of the percolation probability for site and bond percolation on directed square and honeycomb lattices. For the square bond problem we have extended the series from 41 terms to 54, for the square site problem from 16 terms to 37, and for the honeycomb bond problem from 13 terms to 36. Analysis of the series clearly shows that the critical exponent β is the same for all the problems, confirming expectations of universality. For the critical probability and exponent we find in the square bond case, $q_c = 0.355\,299\,4 \pm 0.000\,001\,0$, $\beta = 0.276\,43 \pm 0.000\,10$; in the square site case $q_c = 0.294\,515 \pm 0.000\,005$, $\beta = 0.2763 \pm 0.0003$; and in the honeycomb bond case $q_c = 0.177\,143 \pm 0.000\,002$, $\beta = 0.2763 \pm 0.0002$. In addition we have obtained accurate estimates for the critical amplitudes. In all cases we find that the leading correction to scaling term is analytic, i.e. the confluent exponent $\Delta = 1$.

1. Introduction

Directed percolation (DP) was originally introduced by Broadbent and Hammersley (1957) as a model of fluid flow through a random medium and has since been associated with a wide variety of physical processes. In *static* interpretations, the preferred direction is a spatial direction, and DP could represent the percolation of fluid through porous rock with a certain fraction of the channels blocked (De'Bell and Essam 1983b), crack propagation (Kertész and Viscek 1980) or electric current in a diluted diode network (Redner and Brown 1981). In *dynamical* interpretations, the preferred direction is time, and DP is modelled by a stochastic cellular automaton (Kinzel 1985) in which all lattice sites evolve simultaneously and the main interpretation is as an epidemic without immunization (Harris 1974, Liggett 1985). The behaviour of these models is generally controlled by a single parameter p, which could be the probability that a channel is open or the infection probability, depending on one's favourite interpretation.

When p is smaller than a critical value p_c, the fluid does not percolate through the rock (the epidemic dies out). Let $P(p)$ be the probability that the wetted region percolates infinitely far from the source (the ultimate survival probability in epidemic language) then one expects:

$$P(p) \propto (p - p_c)^{\beta} \qquad p \to p_c^{+}. \tag{1}$$

DP-type transitions are also encountered in many other situations, perhaps most prominently in Reggeon field theory (Grassberger and Sundermeyer 1978, Cardy and Sugar

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1980), chemical reactions (Schlögl 1972, Grassberger and de la Torre 1979), in numerous models for heterogeneous catalysis and surface reactions (Ziff *et al* 1986, Köhler and ben-Avraham 1991, Zhuo *et al* 1993, Jensen 1994), self-organized criticality (Obukhov 1990, Paczuski *et al* 1994) and even galactic evolution (Schulman and Seiden 1982). This short and far from complete list clearly demonstrates that directed percolation is a problem which emerges in a diverse set of physical problems and therefore deserves a great deal of attention.

In this paper we discuss series expansions for the percolation probability on directed square and honeycomb lattices. The earliest series expansion for the square bond problem was the eight terms calculated by Blease (1977). A great improvement was due to Baxter and Guttmann (1988) who extended this series to 41 terms. For the honeycomb bond problem, Onody (1990) obtained a 13-term series and for the square site problem the longest series of 16 terms is due to Onody and Neves (1992) improving the previous record of 10 terms held by De'Bell and Essam (1983a). Using the finite-lattice method pioneered in this context by Baxter and Guttmann (1988) we have extended these series to 54 terms for the square bond problem, 37 terms for the square site problem and 36 terms for the honeycomb bond problem. The percolation probability for the honeycomb site problem is related very simply to that of the square site problem, $P^{HC}(p) = P^{SQ}(p^2)$ (Dhar *et al* 1982, Essam and De'Bell 1982). Note also that bond percolation on the honeycomb lattice may be viewed as site-bond percolation on the square lattice (Essam and De'Bell 1982). In passing, we note that long series have been obtained for the moments of the pair connectedness for the site and bond problems on square and triangular lattices (Essam *et al* 1986, 1988).

2. The finite-lattice method

We wish to calculate the series expansion of the percolation probability on square and honeycomb lattices oriented as in figure 1. We shall consider both site and bond percolation on these lattices. In site (bond) percolation each site (bond) is independently present with probability p and absent with probability $q = 1 - p$. Two sites are connected if one can find a path passing through occupied sites (bonds) only, while always following the allowed directions. For an infinite system, when q is less than a critical value q_c, there is an infinite cluster spanning the lattice. The order parameter of the system is the percolation probability $P(q)$, i.e. the probability that a given site belongs to the infinite cluster. Note that a path passing through a given site can only lead to the sites shown in figure 1 below the origin O. This naturally leads one to consider a finite-lattice approximation to $P(q)$, namely the probability $P_N(q)$ that the origin is connected to at least one site in the Nth row. $P_N(q)$ is a polynomial in q with integer coefficients and a maximal order determined by the total number of sites (bonds) that may be present on the finite lattice.

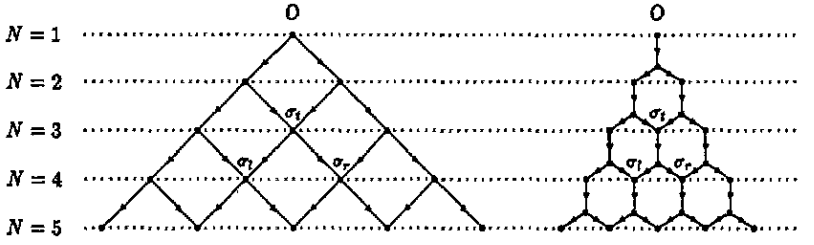

Figure 1. The directed square and honeycomb lattices with orientation given by the arrows. The rows are labelled according to the text.

It has been proved (Bousquet-Mélou 1995), for all the problems considered in this paper, that the polynomials $P_N(q)$ have a formal limit in the algebra of formal power series in the variable q, and therefore $P(q) = \lim_{N\to\infty} P_N(q)$. In all cases one finds that $P_N(q)$ converges to $P(q)$ in such a way that the first N (or $N-1$ depending on the specific problem) terms of the polynomials $P_N(q)$ coincide with those of $P(q)$.

2.1. *Specification of the models*

In order to calculate $P_N(q)$ we associate a state σ_j with each site, such that $\sigma_j = 1$ if site j is connected to the Nth row and $\sigma_j = -1$ otherwise. We shall often write $+/-$ for simplicity. Note that a site can be in state -1 even though, in the case of site percolation, it is itself occupied, or, in the case of bond percolation, bonds emanating from the site are present. Let l, r denote the sites below t as in figure 1. We then define the triangle weight function $W(\sigma_t|\sigma_l, \sigma_r)$ as the probability that the top site t of the triangle is in state σ_t, given that the lower sites l to the left and r to the right are in states σ_l and σ_r, respectively. One can then prove (Bidaux and Forgacs 1984, Baxter and Guttmann 1988) that

$$P_N(q) = \sum_{\{\sigma\}} \prod_t W(\sigma_t|\sigma_l, \sigma_r) \tag{2}$$

where the product is over all sites j of the lattice above the Nth row. The sum is over all values ± 1 of each σ_t, other than the topmost spin σ_1 which always takes the value $+1$. The spins in the Nth row are fixed to be $+1$. In short $P_N(q)$ is calculated as the sum over all possible configurations of the probability of each individual configuration.

Table 1. The triangle weight functions for the various directed percolation problems. Generally one has $W(-|\sigma_l, \sigma_r) = 1 - W(+|\sigma_l, \sigma_r)$.

Problem	$W(+\|+,+)$	$W(+\|+,-) = W(+\|-,+)$	$W(+\|-,-)$
SQ-bond	$1-q^2$	$1-q$	0
HC-bond	$(1-q)(1-q^2)$	$(1-q)^2$	0
SQ-site	$1-q$	$1-q$	0
HC-site	$(1-q)^2$	$(1-q)^2$	0

The weights W are listed in table 1. Obviously, $W(-|\sigma_l, \sigma_r) = 1 - W(+|\sigma_l, \sigma_r)$. The remaining weights are easily calculated by considering the various possible arrangements of states and bonds. $W(+|-,-) = 0$ because the top site is connected to the Nth row if and only if at least one of the neighbours is connected. Let us next look at the remaining square bond weights. $W(+|+,+) = 1-q^2$ because the only bond configuration *not* allowed is both bonds absent which has probability q^2. Finally, $W(+|+,-) = W(+|-,+) = 1-q$ because the bond connecting the two $+$ states has to be present, which happens with probability $p = 1-q$, and the other bond can be either present or absent. For the honeycomb bond problem we find that $W^{HC}(+|\sigma_l, \sigma_r) = (1-q)W^{SQ}(+|\sigma_l, \sigma_r)$ because if the top state is $+1$ the vertical bond has to be present. Note that one can think of the honeycomb bond problem as site-bond percolation on the directed square lattice where both sites and bonds are present with equal probability (Essam and De'Bell 1982). For the square site problem the weights are a little simpler since a site can be in state $+1$ only if it is present and W picks up only the probability of the top state, therefore $W(+|-,-) = 0$ as before and all the other weights with a $+1$ top state are equal to $1-q$. The honeycomb site weights are derived from the square site weights in the same manner as for the bond case. Note

that it is customary to assume in site percolation problems that the origin is present with probability 1.

For the square and honeycomb site problems we therefore find

$$P_N(q) = \sum_{\{\sigma\}} \prod_t W(\sigma_t|\sigma_l, \sigma_r) = \sum_{\{\sigma\}} W_O(0|\sigma_2, \sigma_3) \prod_t W(\sigma_t|\sigma_l, \sigma_r) .$$

The weights $W(\sigma_t|\sigma_l, \sigma_r)$ are those of table 1 and W_O is the weight of the topmost triangle. It is clear from table 1 that for the *site* problem $W^{HC}(q) = W^{SQ}(2q - q^2)$. Since the 'top' weights are 1 for the square site problem and $1 - q$ for the honeycomb site problem we find that

$$P_N^{HC}(q) = (1 - q)P_N^{SQ}(1 - (1 - q)^2) \tag{3}$$

which is essentially the relation mentioned in the introduction, derived from the work of Dhar *et al* (1982) by Essam and De'Bell (1982).

2.2. *Series expansion algorithm*

For small N it is quite easy to calculate $P_N(q)$ by hand, but for larger N one obviously has to resort to computer algorithms. The algorithms are basically implementations of a transfer-matrix method. From (2) we see that the evaluation of $P_N(q)$ involves only local 'interactions' since the weights involve only three neighbouring sites. The sum over all configurations can therefore be performed by moving a boundary line through the lattice. At any given stage this line cuts through a number of, say m, lattice sites thus leading to a total of 2^m possible configurations along this line. Any configuration along the line is trivially represented as a binary number by letting the rth bit of the number equal $(\sigma_r + 1)/2$. For each configuration along the boundary line one maintains a (truncated) polynomial which equals the sum of the product of weights over all possible states on the side of the boundary already traversed. The boundary is moved through the lattice one site at a time. In figure 2 we show how the boundary is moved in order to pick up the weight associated with a given triangle at position r along the boundary line. Let $S0 = (x_1, \ldots, x_{r-1}, 0, x_{r+1}, \ldots, x_m)$ be the configuration of sites along the boundary with 0 at position r and similarly $S1 = (x_1, \ldots, x_{r-1}, 1, x_{r+1}, \ldots, x_m)$ the configuration with 1 at position r. Then in moving the rth site from the bottom left to the top of the triangle we see that the polynomials associated with these configurations are updated as

$$\begin{aligned} P(S0) &= W(0|0, x_{r-1})P(S0) + W(0|1, x_{r-1})P(S1) \\ P(S1) &= W(1|0, x_{r-1})P(S0) + W(1|1, x_{r-1})P(S1) . \end{aligned} \tag{4}$$

The calculation of $P_N(q)$ by this method is limited by memory, since one needs storage for 2^{N-1} boundary configurations. To alleviate this problem one can introduce a cut into the lattice, fix the states on this cut, evaluate the lattice sum $P_N^C(q)$ for each configuration C of the cut, and finally get $P_N(q) = \sum_C P_N^C(q)$ as the sum over all configurations of the cut. By placing the cut appropriately, the growth in memory requirements can be reduced to $2^{N/2}$. Obviously the finite-lattice calculation for different configurations of the cut are independent of one another and these algorithms are therefore perfectly suited to take full advantage of modern massively parallel computers. In the following section we give a few more details of the algorithms we have used.

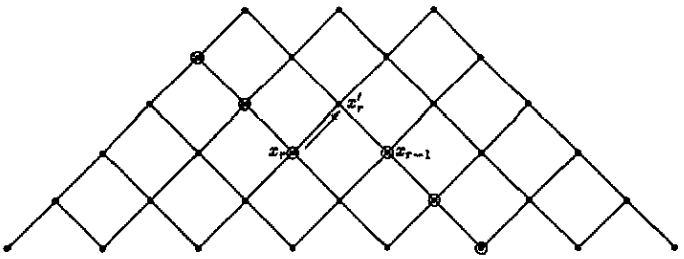

Figure 2. Part of the directed square lattice with the present boundary indicated by open circles. All weights to the left of this boundary have been summed. The weight of the triangle given by (x_r, x'_r, x_{r-1}) is picked up by moving the boundary from x_r to x'_r and updating the associated polynomials according to equation (4).

2.2.1. The bond-problem algorithm. A very efficient algorithm was devised by Baxter for the square bond problem (Baxter and Guttmann 1988). All we had to do for the present work was basically to parallelize the algorithm in order to fully utilize the Intel Paragon at Melbourne University. The algorithm is based on an ingenious transformation of the square bond problem onto a honeycomb lattice. This is done by noting that the square bond weights can be written as

$$W(\sigma_t|\sigma_l, \sigma_r) = \sum_{\sigma_m=\pm 1} f(\sigma_t, \sigma_m) g(\sigma_l, \sigma_m) g(\sigma_r, \sigma_m) \tag{5}$$

where

$$\begin{aligned} &f(+,+) = -1 \qquad f(+,-) = f(-,+) = 1 \qquad f(-,-) = 0 \\ &g(+,+) = q \qquad g(+,-) = g(-,+) = g(-,-) = 1\,. \end{aligned} \tag{6}$$

This means that if we replace each upwards-pointing triangle in figure 1 by a three-pointed star, arriving at the honeycomb lattice of figure 3, then $P_N(q)$ can be calculated from this lattice by assigning weights $f(\sigma_i, \sigma_j)$ to vertical edges (i, j) and $g(\sigma_i, \sigma_j)$ to non-vertical edges.

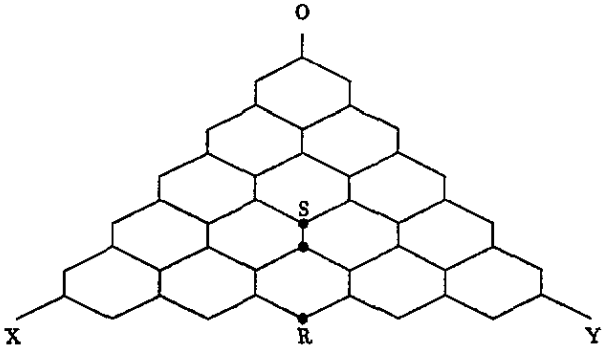

Figure 3. The transformed lattice used in the square bond algorithm. The sites marked with the full circles on the cut-line SR are fixed.

A cut of length L is introduced along the line RS in figure 3 and the transfer-matrix technique is used to build up the lattice to the left of RSO, starting from XR and working upwards to OS. The lattice is symmetrical around the central axis RSO and one can therefore

obtain the lattice sum for the whole lattice by forming the sum of the squares of each boundary line polynomial. After this operation, the whole lattice is summed except for the edges on the centre-line. So finally one has to multiply the (squared) boundary line polynomials by the weights of these edges. This is where the great advantage of the transformation becomes clear. Because $f(-,-)=0$ we need never consider configurations of the cut (or parts of the boundary line in the vertical position) which have any $(-,-)$ edge. This basically means that the number of configurations of the cut which contribute to $P_N(q)$ grow only like $3^{L/2}$ rather than the usually expected 2^L. The transformation thus provides us with an exponentially faster algorithm. Likewise, as parts of the boundary line enter the vertical position, no $(-,-)$ edges need to be considered, which leads to a significant reduction in the length of the cut for a given amount of memory. The memory requirement for the algorithm is governed by the maximal extent of the boundary line, which is at XR, and hence grows like $2^{N/2-1}$. With this algorithm we calculated $P_N(q)$ for $N \leqslant 39$. Since the integer coefficients occuring in the series expansion become very large the calculation was performed using modular arithmetic (see, for example, Knuth 1969). Each run, using a different modulus, took approximately 24 hours using 50 nodes on an Intel Paragon.

Virtually the same algorithm can be used for the honeycomb bond or site-bond square problem except that the f weights have to be replaced by

$$f(+,+)=-(1-q) \qquad f(+,-)=f(-,+)=1-q \qquad f(-,-)=q\,. \tag{7}$$

Since $f(-,-)$ no longer equals 0, obviously the great advange of the original transformation vanishes and the number of configurations of the cut grow like 2^L. For this reason we had to stop calculating $P_N(q)$ at $N=33$, where each modulus required about 32 hours of CPU time using 50 nodes.

2.2.2. The site-problem algorithm. For the site problem the growth in memory can be limited to $2^{N/2-1}$ by introducing a cut across the lattice at row $N/2$. The upper part of the lattice is built up first by the transfer-matrix technique, yielding a partial lattice sum P_U^C and then the lower part P_L^C is done. The total lattice sum for a given cut P_N^C is simply the product of these, i.e. $P_N^C = P_L^C P_U^C$. Again $P_N(q)$ is the sum over all configurations of the cut. It might seem that the numbers of cuts grow as $2^{N/2}$. Substantial simplifications can, however, be obtained. Note first of all that there is symmetry around the central vertical line, which basically reduces the number of cut-configurations by a factor of 2. A more subtle means of reducing the number of cuts is obtained as follows. Since all triangle weights with at least one + on the bottom are the same, it follows that for any two configurations C and C', which can be turned into one another by changing any number of +s to −s *without* adding or removing any $(--)$ sequences, $P_U^C = P_U^{C'}$. This means, for example, that for any cut C without $(--)$ occurences, P_U^C equals the partial sum of the all +s cut. It is possible to use this property to perform the lower lattice sum simultaneously for many cuts. As an example consider cuts starting with $++$ and $-+$ but which otherwise are the same. The upper part is the same and the lower part is also the same except for the weight of the left-most triangle on the cut. By considering the various possibilities when moving the boundary line across this point, one can easily see that the two configurations can be summed simultaneously, i.e. the $-+$ cut can be made as part of the $++$ cut. We calculated $P_N(q)$ for $N \leqslant 32$, which took about 48 hours for each modulus with $N=32$ using 50 nodes.

We also calculated the series expansion for the honeycomb site problem up to $P_{32}(q)$. Although we know the exact relation between the two site problems (3), this calculation

provides us with an extra check of the algorithm and the extrapolation formulae we shall discuss presently.

3. Extrapolation of the series

As mentioned $P_N(q)$ will generally agree with the series for $P(q)$ up to some order determined by N. For the square bond problem the coefficients of $P_N(q) = \sum_{m\geqslant 0} a_{N,m}q^m$ agree with those of $P(q) = \sum_{m\geqslant 0} a_m q^m$ to order N. Baxter and Guttmann (1988) found that the series for $P(q)$ can be extended considerably by determining the correction terms to $P_N(q)$. Let us look at

$$P_N - P_{N+1} = q^N \sum_{r\geqslant 1} q^r d_{N,r}\,. \tag{8}$$

Then we shall call $d_{N,r} = a_{N,N+r} - a_{N+1,N+r}$ the rth correction term. Obviously if one can find formulae for $d_{N,r}$ for all $r \leqslant K$ then one can use the series coefficients of $P_N(q)$ to extend the series for $P(q)$ to order $N + K$ since

$$a_{N+k} = a_{N,N+k} - \sum_{m=1}^{k} d_{N+k-m,m} \tag{9}$$

for all $k \leqslant K$. That this method can be very efficient was clearly demonstrated by Baxter and Guttmann, who identified the first 12 correction terms and used $P_{29}(q)$ to extend the series for $P(q)$ to 41 terms. To really appreciate this advance one should bear in mind that the time it takes to calculate $P_N(q)$ grows exponentially with N, so a direct calculation correct to the same order would have taken years rather than days. In the following we will give details of the correction terms for the various cases.

3.1. The square bond case

The first correction term for the square bond case is given by the Catalan numbers

$$d_{N,1} = c_N = (2N!)/N!(N+1)! \tag{10}$$

a result which was proved (Bousquet-Mélou 1995) by noting that the correction term arises from compact bond animals of directed height N and perimeter $N+1$. The second correction term

$$d_{N,2} = 2c_N - c_{N+1} \tag{11}$$

was also calculated exactly recently (Bousquet-Mélou 1995). As noted by Baxter and Guttmann (1988) the higher-order correction terms $d_{N,r}$ can be expressed as rational functions of the Catalan numbers. We have found that $d_{N,r}$ can always be written in the form

$$d_{N,r} = \sum_{k=1}^{[(r-1)/2]} A_{r,k} \binom{N-m}{k} c_{N-m} + \sum_{j=1}^{2r-4} B_{r,j} c_{N-r+2+j} \tag{12}$$

where $m = \max(0, r-4-2k)$. These formulae hold for all available N, provided that only Catalan numbers c_m with $m \geqslant 0$ are involved. As noted by Baxter and Guttmann it is also true for $m = -1$ provided one 'defines' $c_{-1} = -1$ (there was a misprint at this point in the original article). Thus the extrapolation formulae are true for $N \geqslant r-4$. For $r \leqslant 15$ the coefficients $A_{r,k}$ and $B_{r,j}$ are either integers or fractions with small (two or five) denominators. Note that there are various relations between the Catalan numbers so

there are infinitely many ways of writing (12). For several of the correction formulae the general form adopted in this paper is slightly different from that of Baxter and Guttmann (1988) who tried wherever possible to choose a form involving only integers. The trade-off for having a general expression for the correction terms is that more rational fractions become involved. However, with the proliferation of powerful mathematical packages such as MAPLE and MATHEMATICA this trade-off is well worth while. In table 2 we have listed the coefficients $A_{r,j}$ and $B_{r,j}$ for $r \leqslant 15$. Using these extrapolation formulae and the series for $P_{39}(q)$ we have extended the series for $P(q)$ to the 54 terms given in table 3.

3.2. *The square site case*

Inspired by the success of the extrapolation procedure for the square bond problem one might hope for similar success for other problems. And indeed one can find several of the correction terms for the square site problem, although the success is less spectacular as one is restricted to the first six correction terms. The first correction term $d_{N,1}$ was identified by Onody and Neves (1992) and has since been computed exactly by Bousquet-Mélou (1995):

$$d_{N,1} = \frac{(3N)!}{N!(2N+1)!}\,. \tag{13}$$

This expression for the correction term was identified by Onody and Neves (1992) as the number of ways of inserting $n-4$ sheets through a ball having n vertices on its surface such that pairs of sheets meet only on surface curves joining vertices! While this is true, a more useful and pertinent interpretation can be given. Viennot (1994) has pointed out that this is just the expression for the number of ternary trees of n vertices, which in turn is isomorphic to the number of diagonally convex directed animals (Delest and Fédou 1989). It is the identification between these animals and the first correction term that has been proved by Bousquet-Mélou (1995). She also proved our formula for the second correction term.

As in the square bond case we can express higher correction terms as a function of $d_{N,1}$. Again, there are infinitely many ways of expressing the formulae for the correction terms, one of which is

$$d_N^r = \sum_{i=2}^{r-1} C_{r,i} \binom{N}{i} d_{N,1} + \sum_{j=1}^{2r-1} (NB_{r,j} + A_{r,j}) d_{K,1} \tag{14}$$

where $K = N - r + j$ for $r \leqslant 4$ and $K = N - r - 1 + j$ for $r \geqslant 5$. These formulae are correct up to $r = 6$, whenever $N \geqslant r$. The coefficients are listed in table 4. These formulae allow us to extend the series for $P(q)$ by an additional six terms to a total of 37 terms listed in table 5.

3.3. *The honeycomb bond case*

For bond percolation on the directed honeycomb lattice Bousquet-Mélou (1995) proved that the generating function $f = \sum_{N \geqslant 1} d_{N,1} t^{N-1}$ of the first correction term $d_{N,1}$ is characterized by the algebraic equation

$$f = 1 + tf + \frac{t}{4}((7+t)f^2 + f^3)\,. \tag{15}$$

The higher-order correction terms are given by the formulae

$$d_N^r = \sum_{k=1}^{r} \left[D_{r,k} \binom{N}{3} + C_{r,k} \binom{N}{2} \right] d_{N-2r+2+k,1} + \sum_{j=1}^{2r} (NB_{r,j} + A_{r,j}) d_{N-2r+2+j,1} \tag{16}$$

Table 2. The coefficients $A_{r,m}$ and $B_{r,j}$ in the extrapolation formulae for the square bond problem.

									r				
j	3	4	5	6	7	8	9	10	11	12	13	14	15
1	−2	2	2	6	28	168	1188	9438	81796	758472	7423832	75951512	806254512
2	2	−5	13	14	40	172	940	6022	43224	337852	2821008	24820084	227759640
3		5	$-29\frac{1}{2}$	58	76	230	916	4358	22626	116670	490220	−39210	−43047748
4		−2	−2	−73	218	−23	16720	−238382	5439166	$-54821200\frac{3}{5}$	$1211505650\frac{4}{5}$	$-8785667878\frac{4}{5}$	$228722521818\frac{2}{5}$
5			12	24	−360	1185	−2285	66470	−1329734	$13844727\frac{2}{5}$	$-302226767\frac{1}{5}$	$2199183639\frac{1}{5}$	$-57173632471\frac{3}{5}$
6			−2	−6	174	−1619	3737	20838	−289156	3246796	−69778186	503746552	−13371140646
7				0	−128	889	−7985	23543	−67332	$1144926\frac{1}{5}$	$-20852206\frac{3}{5}$	$165626416\frac{3}{5}$	$-4004317118\frac{4}{5}$
8				0	82	−242	4320	−35091	56180	355092	−4943452	44277656	−999749068
9					−15	−15	−1334	23956	−177605	$406391\frac{4}{5}$	$-1134497\frac{2}{5}$	$12747113\frac{2}{5}$	$-251625122\frac{1}{5}$
10					0	18	−104	−8722	124506	$-793053\frac{3}{5}$	$1126018\frac{4}{5}$	$3369765\frac{1}{5}$	$-63360349\frac{3}{5}$
11						−1	243	2820	−56441	675899	−3956056	7537547	−18025630
12						0	−46	−1204	18304	−329150	3495696	−18307058	28433480
13							−2	367	−5702	115282	−1913164	18478108	−90869764
14							0	−44	1636	−25924	715656	−10857182	95785220
15								2	−250	954	−200132	4463999	−61558076
16								0	8	1036	42000	−1380082	27374696
17									−2	−125	−6857	363300	−9209555
18									0	−12	1064	−95182	2442068
19										2	−163	21166	−518177
20										0	12	−2554	86634
21											−2	116	−11594
22											0	−12	1316
23												2	−140

Table 2. Continued.

									r				
j	3	4	5	6	7	8	9	10	11	12	13	14	15
24												0	12
25													−2
26													0
m	3	4	5	6	7	8	9	10	11	12	13	14	15
1	−2	3	$-9\frac{1}{2}$	13	−90	−745	31 152	−485 988	10 837 718	$-109\,826\,658\frac{1}{5}$	$2\,422\,283\,961\frac{3}{5}$	$-17\,572\,113\,989\frac{3}{5}$	$457\,481\,237\,152\frac{4}{5}$
2			4	−2	15	−5	51	3 566	−57 572	$854\,505\frac{3}{5}$	$-13\,937\,224\frac{4}{5}$	$130\,944\,690\frac{4}{5}$	$-2\,410\,518\,810\frac{2}{5}$
3					−8	4	−24	−19	24	$-9\,414\frac{2}{5}$	$101\,155\frac{1}{5}$	$-1\,480\,387\frac{1}{5}$	$19\,377\,361\frac{3}{5}$
4							16	0	40	$81\frac{3}{5}$	$-136\frac{4}{5}$	$21\,604\frac{4}{5}$	$-168\,086\frac{2}{5}$
5									−32	−16	−72	−232	216
6											64	64	144
7													−128

Table 3. The coefficients a_n in the series expansion of $P(q) = \sum_{n\geqslant 0} a_n q^n$ for directed-bond percolation on the square lattice.

n	a_n	n	a_n
0	1	28	−16 161 597 987
1	0	29	−43 448 897 414
2	−1	30	−117 083 094 891
3	−2	31	−315 709 399 172
4	−4	32	−853 195 535 637
5	−8	33	−2 306 601 710 190
6	−17	34	−6 249 350 665 825
7	−38	35	−16 933 569 745 596
8	−88	36	−45 982 825 444 918
9	−210	37	−124 847 185 166 968
10	−511	38	−339 715 065 397 631
11	−1 264	39	−923 984 791 735 474
12	−3 165	40	−2 518 902 151 116 767
13	−8 006	41	−6 861 776 192 406 434
14	−20 426	42	−18 738 381 486 019 497
15	−52 472	43	−51 115 047 622 373 452
16	−135 682	44	−139 811 976 659 987 636
17	−352 562	45	−381 836 043 069 041 990
18	−920 924	46	−1 046 008 104 766 969 784
19	−2 414 272	47	−2 859 625 985 546 910 846
20	−6 356 565	48	−7 845 284 416 715 093 642
21	−16 782 444	49	−21 465 842 456 693 034 778
22	−44 470 757	50	−58 976 491 160 296 065 655
23	−118 090 648	51	−161 476 439 366 532 026 854
24	−314 580 062	52	−444 296 183 371 760 430 967
25	−839 379 548	53	−1 217 055 970 699 512 453 538
26	−2 245 969 278	54	−3 353 766 967 706 302 949 866
27	−6 017 177 104		

which we find to be correct for $r \leqslant 4$ and $N \geqslant 2r - 1$. The coefficients are listed in table 6 apart from the Ds since the only non-zero ones are $D_{4,1} = -157\,281/5$ and $D_{4,2} = 1\,744\,273/5$. The final 36-term series for $P(q)$ is given in table 7.

4. Analysis of the series

We expect that the series for the percolation probability behaves like

$$P(q) \sim A(1 - q/q_c)^{\beta}[1 + a_{\Delta}(1 - q/q_c)^{\Delta} + \cdots] \tag{17}$$

where A is the critical amplitude, Δ is the leading confluent exponent and the '$\cdots$' represents higher-order correction terms. By universality we expect β to be the same for all the percolation problems studied in this paper and we will argue that the dominant correction term is analytic, i.e. $\Delta = 1$.

In the following sections we present the results of our analysis of the series which include accurate estimates for the critical parameters q_c, β, A and Δ. For the most part the best results are obtained using Dlog Padé (or in some cases just ordinary Padé) approximants. A comprehensive review of these and other techniques for series analysis may be found in Guttmann (1989).

Table 4. The coefficients $A_{r,j}$, $B_{r,j}$ and $C_{r,j}$ in the extrapolation formulae for the square site problem.

	$A_{r,j}$					$B_{r,j}$					$C_{r,j}$			
r/j	2	3	4	5	6	2	3	4	5	6	3	4	5	6
1	−6	−31	−353	−1 050	−87 304	9	18	126	225	15 372	$3\frac{13}{36}$	$8\frac{1}{81}$	$22\frac{1\,037\,831}{1\,679\,616}$	$56\frac{32\,726\,159}{30\,699\,264}$
2	$8\frac{2}{3}$	11	−199	$-5\,458\frac{1}{3}$	−71 645	$46\frac{1}{3}$	$-12\frac{2}{3}$	$120\frac{2}{3}$	$1\,388\frac{2}{3}$	$13\,474\frac{2}{3}$		$-6\frac{35}{216}$	$-12\frac{10\,969}{15\,552}$	$-36\frac{2\,734\,421}{5\,038\,848}$
3	−12	$-73\frac{3}{4}$	−123	$-2\,965\frac{2}{3}$	−85 165	$-7\frac{1}{3}$	$-277\frac{5}{12}$	$150\frac{7}{27}$	1 113	$23\,201\frac{1}{3}$			$11\frac{385}{1296}$	$19\frac{30\,595}{46\,656}$
4		$23\frac{3}{4}$	$249\frac{1}{2}$	$-530\frac{74}{81}$	$-7\,647\frac{61}{81}$		$-8\frac{11}{12}$	$948\frac{17}{108}$	$267\frac{2}{9}$	$2\,194\frac{5}{27}$				$-20\frac{5531}{7776}$
5		$18\frac{1}{3}$	904	$293\frac{2014}{2187}$	$-5\,605\frac{1120}{2187}$		$7\frac{1}{3}$	$862\frac{1}{4}$	$-404\frac{112}{243}$	$3\,084\frac{371}{729}$				
6			$-396\frac{1}{6}$	$-499\frac{193}{256}$	$134\frac{11\,888}{39\,049}$			$-149\frac{5}{12}$	$-1\,524\frac{353\,035}{559\,872}$	$-888\frac{17\,965}{19\,683}$				
7			0	$-8\,771\frac{183}{256}$	$-1\,225\frac{169}{1536}$			0	$-7\,324\frac{35}{256}$	$-5\,593\frac{6\,619\,423}{30\,233\,088}$				
8				$1525\frac{47}{96}$	$39\,744\frac{107}{512}$				$362\frac{175}{192}$	$31\,048\frac{1321}{1536}$				
9				392	$20\,245\frac{19}{192}$				112	$10\,870\frac{37}{48}$				
10					$-8\,251\frac{1}{192}$					$-2\,273\frac{107}{192}$				
11					0					0				

Table 5. The coefficients a_n in the series expansion of $P(q) = \sum_{n\geqslant 0} a_n q^n$ for directed site percolation on the square lattice.

n	a_n	n	a_n
0	1	19	−92 459 524
1	0	20	−298 142 956
2	−1	21	−922 424 269
3	−3	22	−3 098 690 837
4	−8	23	−9 042 937 179
5	−21	24	−34 187 149 573
6	−56	25	−79 544 646 085
7	−154	26	−439 149 878 359
8	−434	27	−313 237 196 088
9	−1 252	28	−7 786 443 675 714
10	−3 675	29	16 637 473 844 344
11	−10 954	30	−207 593 240 544 002
12	−33 044	31	973 714 665 769 453
13	−100 676	32	−7 311 741 153 076 579
14	−309 569	33	43 345 744 201 832 502
15	−957 424	34	−292 472 879 532 946 388
16	−2 987 846	35	1 867 850 225 746 155 582
17	−9 330 274	36	−12 389 925 641 797 917 900
18	−29 522 921	37	81 441 868 912 809 214 904

Table 6. The coefficients $A_{r,j}$, $B_{r,j}$ and $C_{r,j}$ in the extrapolation formulae for the honeycomb bond problem.

	$A_{r,j}$			$B_{r,j}$			$C_{r,j}$		
r/j	2	3	4	2	3	4	2	3	4
1	$-8\frac{3}{20}$	$10\,954\frac{1}{2}$	$-2\,773\,464\frac{741}{1000}$	$14\frac{1}{4}$	$-5\,696\frac{41}{50}$	$814\,389\frac{147}{1000}$	$-12\frac{13}{15}$	$1\,753\frac{9}{25}$	$-86\,965\frac{91}{125}$
2	$-67\frac{13}{20}$	$87\,495\frac{21}{50}$	$-24\,330\,909\frac{122}{125}$	$54\frac{2}{5}$	$-48\,502\frac{51}{100}$	$7\,805\,538\frac{1733}{2000}$	$-141\frac{8}{15}$	$19\,295\frac{4}{25}$	$-1\,118\,660\frac{159}{500}$
3	$-3\frac{3}{5}$	$1\,831\frac{14}{25}$	$415\,040\frac{911}{1000}$	$4\frac{9}{20}$	$-292\frac{1}{20}$	$-424\,945\frac{137}{2000}$	$12\frac{13}{15}$	$-1\,663\frac{19}{25}$	$224\,372\frac{37}{250}$
4		$593\frac{3}{10}$	$-276\,663\frac{5}{8}$		$-616\frac{9}{20}$	$98\,675\frac{69}{80}$			
5		$66\frac{13}{20}$	$-56\,787\frac{1}{5}$		$23\frac{1}{2}$	$31\,375\frac{77}{80}$			
6		$7\frac{4}{5}$	$-8\,377\frac{23}{40}$		$5\frac{1}{5}$	$9\,476\frac{1}{2}$			
7			$-646\frac{1}{4}$			$871\frac{19}{40}$			
8			$-315\frac{3}{5}$			$-187\frac{9}{10}$			

4.1. q_c and β

In table 8 we show the Dlog Padé approximants to the percolation probability series for bond percolation on the directed square lattice. The defective approximants, those for which there is a spurious singularity on the positive real axis closer to the origin than the physical critical point, are marked with an asterisk. The overwhelming majority of the approximants cluster around the values $q_c = 0.355\,299\,4$ and $\beta = 0.276\,43$. As always in this type of analysis it is very difficult to accurately judge the true errors of the estimates of the critical parameters, however we venture to say that the critical parameters lie in the ranges $q_c = 0.355\,299\,4(10)$ and $\beta = 0.276\,43(10)$, where the figures in parentheses indicate the estimated error on the last digits. The other remarkable feature of table 8 is that surprisingly

Table 7. The coefficients a_n in the series expansion of $P(q) = \sum_{n \geq 0} a_n q^n$ for directed bond percolation on the honeycomb lattice.

n	a_n	n	a_n
0	1	19	−1 103 369 168 956
1	−1	20	−5 771 541 600 014
2	−4	21	−31 153 472 926 184
3	−12	22	−160 153 702 442 390
4	−45	23	−907 425 183 546 587
5	−188	24	−4 317 291 410 619 157
6	−835	25	−28 433 248 376 749 141
7	−3 849	26	−99 125 481 158 184 567
8	−18 242	27	−1 076 035 285 073 833 314
9	−88 265	28	−238 091 850 291 444 337
10	−434 295	29	−58 631 611 223 043 405 378
11	−2 165 198	30	279 283 045 229 982 597 450
12	−10 915 089	31	−4 730 770 444 199 592 196 256
13	−55 534 781	32	40 182 669 640 102 878 093 220
14	−284 708 699	33	−480 633 574 529 182 764 438 221
15	−1 470 350 760	34	4 852 667 371 105 928 333 619 923
16	−7 628 363 273	35	−53 829 647 651 783 620 888 423 836
17	−39 878 267 745	36	574 209 696 129 704 803 372 604 206
18	−208 458 228 964	37	

Table 8. Dlog Padé approximants to the percolation series for directed bond percolation on the square lattice.

	$[N-1, N]$		$[N, N]$		$[N+1, N]$	
N	q_c	β	q_c	β	q_c	β
11	0.355 300 0	0.276 45	0.355 303 0	0.276 53	0.355 302 3	0.276 51
12	0.355 301 6	0.276 49	0.355 301 1	0.276 48	0.355 299 7	0.276 44
13	0.355 302 8*	0.276 52*	0.355 300 4	0.276 46	0.355 300 0	0.276 45
14	0.355 299 4	0.276 43	0.355 297 2	0.276 34	0.355 299 5	0.276 43
15	0.355 299 1	0.276 42	0.355 299 4	0.276 43	0.355 299 4	0.276 43
16	0.355 299 4	0.276 43	0.355 299 4	0.276 43	0.355 299 4	0.276 43
17	0.355 299 4	0.276 43	0.355 299 4	0.276 43	0.355 299 7*	0.276 44*
18	0.355 299 4	0.276 43	0.355 299 2	0.276 42	0.355 298 3	0.276 32
19	0.355 300 2*	0.276 43*	0.355 299 1	0.276 41	0.355 299 6*	0.276 44*
20	0.355 299 4	0.276 43	0.355 299 4	0.276 43	0.355 299 4	0.276 43
21	0.355 299 4	0.276 43	0.355 299 4	0.276 43	0.355 299 4*	0.276 43*
22	0.355 299 4*	0.276 43*	0.355 299 4*	0.276 43*	0.355 299 4*	0.276 43*
23	0.355 299 4*	0.276 43*	0.355 299 4	0.276 43	0.355 299 4	0.276 43
24	0.355 299 3*	0.276 43*	0.355 299 3*	0.276 43*	0.355 299 5	0.276 44
25	0.355 299 3*	0.276 43*	0.355 299 7	0.276 45	0.355 299 5*	0.276 44*
26	0.355 299 1*	0.276 43*	0.355 299 0*	0.276 43*	0.355 298 6*	0.276 47*
27	0.355 299 3*	0.276 43*				

many of the high-order approximants are defective.

The results of the analysis of the series for the square site problem are listed in table 9. In this case there is a marked upward drift in the estimates for both q_c and β and the estimates do not settle down to definite values. It does, however, seem likely that the true critical parameters lie within the estimates $q_c = 0.294\,515(5)$ and $\beta = 0.2763(3)$.

Table 9. Dlog Padé approximants to the percolation series for directed site percolation on the square lattice.

	$[N-1, N]$		$[N, N]$		$[N+1, N]$	
N	q_c	β	q_c	β	q_c	β
5	0.293 933 7	0.268 81	0.294 329 1	0.272 66	0.294 297 9	0.272 28
6	0.294 267 0	0.271 90	0.294 317 5	0.272 52	0.294 269 9*	0.271 99*
7	0.294 416 8	0.273 93	0.294 452 1	0.274 53	0.294 477 7	0.275 02
8	0.294 513 5	0.275 85	0.294 474 2	0.274 95	0.294 479 4*	0.275 05*
9	0.294 459 9	0.274 65	0.294 472 0	0.274 90	0.294 473 9	0.274 94
10	0.294 475 3	0.274 98	0.294 465 6*	0.274 78*	0.294 494 2	0.275 46
11	0.294 522 8	0.276 55	0.294 515 6	0.276 23	0.294 502 0	0.275 71
12	0.294 524 6*	0.276 62*	0.294 506 0	0.275 86	0.294 505 8	0.275 85
13	0.294 505 8	0.275 85	0.294 506 1*	0.275 86*	0.294 504 7	0.275 81
14	0.294 505 1	0.275 78	0.294 505 1	0.275 82	*	*
15	0.294 505 6	0.275 84	0.294 504 7*	0.275 81*	0.294 503 2*	0.275 76*
16	0.294 506 9	0.275 89	0.294 509 6	0.276 02	0.294 508 9	0.275 98
17	0.294 509 0	0.275 99	0.294 509 5	0.276 01	0.294 511 3	0.276 12
18	0.294 513 4	0.276 25	0.294 511 1	0.276 11		

Table 10. Dlog Padé approximants to the percolation series for directed bond percolation on the honeycomb lattice.

	$[N-1, N]$		$[N, N]$		$[N+1, N]$	
N	q_c	β	q_c	β	q_c	β
5	0.177 022 9	0.273 31	0.177 072 2	0.274 20	0.177 113 1	0.275 07
6	0.177 119 5	0.275 23	0.177 096 7	0.274 69	0.177 106 7	0.274 93
7	0.177 108 7	0.274 98	0.177 116 1	0.275 17	0.177 127 0	0.275 52
8	0.177 132 0	0.275 72	0.177 020 9*	0.276 62*	0.177 141 4	0.276 12
9	0.177 148 0	0.276 47	0.177 129 4	0.275 59	0.177 136 9	0.275 91
10	0.177 139 1	0.276 01	0.177 135 2	0.275 84	0.177 135 6	0.275 85
11	0.177 135 7	0.275 86	0.177 134 4*	0.275 80*	0.177 139 9	0.276 09
12	0.177 141 2	0.276 19	0.177 138 1	0.275 98	0.177 139 5	0.276 06
13	0.177 140 2	0.276 12	0.177 141 1	0.276 18	0.177 140 3	0.276 12
14	0.177 140 6	0.276 14	0.177 140 4	0.276 41	0.177 140 3*	0.276 12*
15	0.177 140 5	0.276 13	0.177 140 8	0.276 16	0.177 142 9	0.276 36
16	0.177 139 0*	0.276 05*	0.177 141 5	0.276 22	0.177 141 9	0.276 25
17	0.177 142 2	0.276 29	0.177 141 8	0.276 24	0.177 141 8	0.276 24
18	0.177 141 8	0.276 24				

The analysis of the series for the honeycomb bond problem yields the results in table 10. Again we see an upward drift in the estimates for both q_c and β though the estimates are somewhat more stable than in the previous case. It seems likely that the true critical parameters lie within the estimates $q_c = 0.177\,143(2)$ and $\beta = 0.2763(2)$.

Finally we analysed the series for the honeycomb site problem, with the results tabulated in table 11. As in the square site case there is a very pronounced upward drift in the estimates for both q_c and β. It seems likely that the true critical parameters lie within the estimates $q_c = 0.160\,067(5)$ and $\beta = 0.2763(4)$. We note that the expected relation between the values of q_c for the square site and honeycomb site problems, $q_c^{SQ} = 2q_c^{HC} - (q_c^{HC})^2$, is clearly fulfilled by the estimates. This inspires some confidence in the appropriateness

Table 11. Dlog Padé approximants to the percolation series for directed site percolation on the honeycomb lattice.

	$[N-1, N]$		$[N, N]$		$[N+1, N]$	
N	q_c	β	q_c	β	q_c	β
5	0.159 815 9	0.270 17	0.159 957 3	0.272 65	0.159 949 1	0.272 49
6	0.159 926 9	0.272 03	0.159 951 6	0.272 54	0.159 948 7*	0.272 48*
7	0.160 018 1	0.274 16	0.160 040 9	0.274 85	0.160 054 5	0.275 32
8	0.160 065 6	0.275 77	0.160 047 6	0.275 07	0.159 968 2	0.271 95
9	0.160 037 8	0.274 73	0.160 045 7	0.275 01	0.160 045 2	0.274 99
10	0.160 045 3	0.274 99	0.160 045 6	0.275 01	0.160 055 5	0.275 43
11	0.160 028 0*	0.274 62*	0.160 071 1	0.276 40	0.160 059 7	0.275 65
12	0.160 049 8	0.275 15	0.160 063 0	0.275 85	0.160 063 0	0.275 85
13	0.160 063 0	0.275 85	0.160 063 0	0.275 85	0.160 062 2	0.275 80
14	0.160 062 0	0.275 79	0.160 062 5	0.275 82	0.160 063 6	0.275 89
15	0.160 063 0	0.275 85	0.160 062 2*	0.275 80*	0.160 039 1*	0.276 65*
16	0.160 064 1	0.275 93	0.160 065 6	0.276 06	0.160 064 7	0.275 97
17	0.160 065 0	0.276 00	0.160 065 5	0.276 04	0.160 066 2	0.276 11
18	0.160 068 8	0.276 42	0.160 066 2	0.276 11		

of our extrapolation method in general and our error estimates in particular.

4.2. The critical amplitudes

From the leading critical behaviour, $P(q) \sim A(1 - q/q_c)^\beta$, it follows that $(q_c - q)P^{-1/\beta}|_{q=q_c} \sim A^{-1/\beta} q_c$. So by forming the series for $G(q) = (q_c - q)P^{-1/\beta}$ we can estimate the critical amplitude A from Padé approximants to G evaluated at q_c. The procedure works well but requires knowledge of both q_c and β. For the square bond series we know both q_c and β very accurately, and we have estimated A using values of q_c between 0.355 299 and 0.3553 and values of β ranging from 0.2764 to 0.2765. For each (q_c, β) pair we calculate A as the average over all $[N + K, N]$ Padé approximants with $K = 0, \pm 1$ and $2N + K \geqslant 45$. The spread among the approximants is minimal for $q_c = 0.355\,299\,4$, $\beta = 0.276\,43$, where $A = 1.329\,147\,5(2)$. Allowing for values of q_c and β within the full range we get $A = 1.3292(5)$.

For the square site series we used values of q_c from 0.294 51 to 0.294 52 and β from 0.2761 to 0.2765, averaging over Padé approximants with $2N + K \geqslant 27$. In this case the spread is minimal for $q_c = 0.294\,515$, $\beta = 0.2763$ with $A = 1.425\,164(5)$. Again allowing for a wider choice of critical parameters we estimate that $A = 1.425(1)$.

For the honeycomb bond series we restricted q_c to lie between 0.177 138 and 0.177 148 and β between 0.2761 to 0.2765, using all approximants with $2N + K \geqslant 26$. The minimal spread occurs at $q_c = 0.177\,143$, $\beta = 0.276\,35$, where $A = 1.106\,07(2)$. A wider choice for q_c and β leads to the estimate $A = 1.106(1)$.

Finally in the honeycomb site case we used values of q_c in the range 0.160 065 to 0.160 075 and β from 0.2761 to 0.2765, using all approximants with $2N + K \geqslant 27$. The minimal spread occurs when $q_c = 0.160\,069$, $\beta = 0.2764$, where $A = 1.167\,79(2)$. With the wider choice of critical parameters we estimate that $A = 1.167(1)$. The exact relation (3) between the square and honeycomb site problems means that there is a simple relation between the amplitudes in the two cases. First note that $A^H(1 - q/q_{c,H})^\beta \sim P^H(q) = (1 - q)P^S(2q - q^2) \sim (1 - q)(1 - (2q - q^2)/q_{c,S})^\beta$. Since $q_{c,S} = 2q_{c,H} - q_{c,H}^2$,

we find that, $(1-(2q-q^2)/q_{c,S})^\beta = [(q_{c,H}-q)(2-q_{c,H}-q)/q_{c,S}]^\beta$, and therefore $A^H = (1-q_{c,H})[(2-2q_{c,H})q_{c,H}/q_{c,S}]^\beta A^S = (1-q_{c,H})(1-q_{c,H}^2/q_{c,S})^\beta A^S$. Insertions of the various critical parameters shows that this relation is indeed satisfied by our amplitude estimates.

A second method, proposed by Liu and Fisher (1989), for calculating critical amplitudes starts by assuming the functional form $P(q) \sim A(q)(1-q/q_c)^\beta + B(q)$. One then transforms this function into $g(q) = (1-q/q_c)^{-\beta}P(q) \sim A(q) + B(q)(1-q/q_c)^{-\beta}$. The required amplitude is now the *background* term in $g(q)$, which can be obtained from inhomogeneous differential approximants (Guttmann 1989, p 89). In table 12 we have listed the estimates obtained by averaging over various first-order differential approximants, using at least 40 terms of the series for the square bond case and at least 25 terms in the other cases. The critical parameters q_c and β, used in the transformation of the series, were the central values of the estimates from the previous section. This method generally yields slightly lower estimates for the amplitudes and the spread among the approximants is much larger than in the first method.

Table 12. Critical amplitudes A for the four percolation problems obtained by using the method of Liu and Fisher (1989). The estimates were calculated by averaging over various inhomogeneous differential approximants of order L.

L	*SQ* bond	*SQ* site	*HC* bond	*HC* site
1	1.29661	1.41614	1.11520	1.16740
2	1.31234	1.39775	1.12002	1.16579
3	1.31114	1.39989	1.12001	1.16607
4	1.31218	1.37739	1.11952	1.16564
5	1.31098	1.39359	1.11750	1.16546
6	1.31006	1.39001	1.11808	1.16521
7	1.32566	1.39889	1.11856	1.16486
8	1.30916	1.39582	1.11929	1.16537
9	1.31322	1.39162	1.11929	1.16534
10	1.31122	1.39449	1.11780	1.16508
11	1.31195	1.40570	1.12056	1.16578
12	1.31228	1.40306	1.12435	1.16462

4.3. The confluent exponent

We studied the series using two different methods in order to estimate the value of the confluent exponent. In the first method, due to Baker and Hunter (1973), one transforms the function $P(q) = \sum_{i=1}^{n} A_i(1-q/q_c)^{-\lambda_i} = \sum_{n=0}^{\infty} a_n q^n$ into an auxiliary function with simple poles at $1/\lambda_i$. We first make the change of variable $q = q_c(1-e^{-\zeta})$ and find, after multiplying the coefficient of ζ^k by $k!$, the auxiliary function

$$\mathcal{F}(\zeta) = \sum_{i=1}^{N}\sum_{k=0}^{\infty} A_i(\lambda_i\zeta)^k = \sum_{i=1}^{N}\frac{A_i}{1-\lambda_i\zeta} \tag{18}$$

which has poles at $\zeta = 1/\lambda_i$ with residue $-A_i/\lambda_i$. The great advantage of this method (when it works) is that one obtains simultaneous estimates for many critical parameters, namely β, Δ and the critical amplitude, while there is only one parameter, q_c, in the transformation. In figure 4 we have plotted, respectively, β and Δ as a function of the transformation parameter q_c for various $[N \pm K, N]$ Padé approximants, with $N \geqslant 25$. For each 'guess'

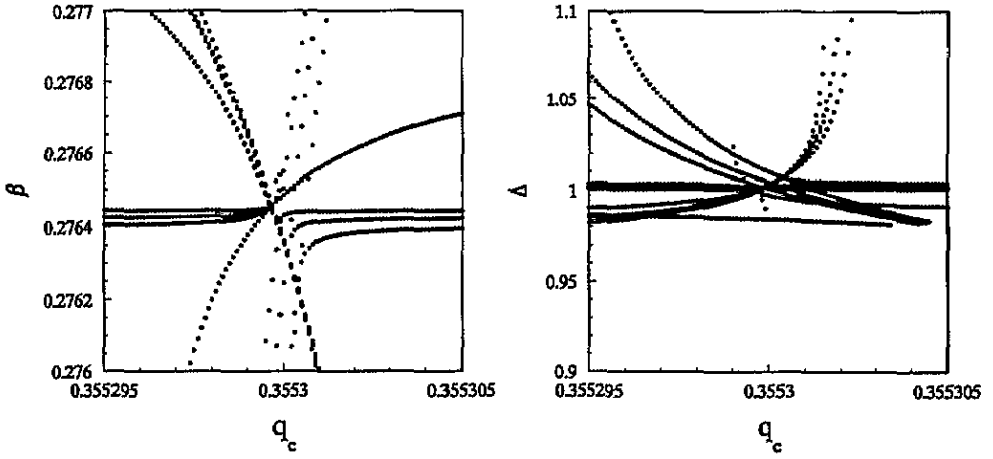

Figure 4. The critical exponent β and confluent exponent Δ as a function of the parameter q_c in the Baker–Hunter transformation for the square bond series.

for q_c, we performed the Baker–Hunter transformation and located the numerically largest and next-largest poles, which are the estimates for the reciprocals of $-\beta$ and $-(\beta + \Delta)$, respectively. The majority of the approximants have a very narrow crossing region close to $q_c = 0.355\,299\,6(3)$, with $\beta = 0.276\,45(3)$ and $\Delta = 1.000(5)$. In table 13 we have listed the estimates for β, Δ and the corresponding critical amplitudes obtained from the Baker–Hunter transformed series with $q_c = 0.355\,299\,6$. The results strongly suggest that the leading correction to scaling term is analytic. Furthermore, we note that the estimates for the critical amplitudes fully agree with those obtained from the first method used in the previous section.

Table 13. Estimates for the critical exponent β, critical amplitude A, confluent exponent Δ and confluent amplitude $A \times a_\Delta$, obtained from $[N, M]$ Padé approximants to the Baker–Hunter transformed square bond series with $q_c = 0.355\,299\,6$.

N	M	β	A	Δ	$A \times a_\Delta$
22	23	0.276 45	1.329 25	1.000 97	1.032 02
23	23	0.276 46	1.329 30	1.000 13	1.030 29
24	23	0.278 63	1.323 69	0.984 39	1.012 24
23	24	0.276 45	1.329 25	1.000 90	1.031 81
24	24	0.276 47	1.329 31	0.999 94	1.029 93
25	24	0.275 49	1.331 00	1.013 75	1.053 22
24	25	0.276 45	1.329 26	1.000 78	1.031 49
25	25	0.276 48	1.329 35	0.999 22	1.028 57
26	25	0.275 89	1.330 38	1.006 98	1.040 48
25	26	0.276 45	1.329 26	1.000 64	1.031 14
26	26	0.276 49	1.329 36	0.999 06	1.028 26
27	26	0.276 17	1.329 92	1.003 05	1.034 10
26	27	0.276 45	1.329 28	1.000 37	1.030 52
27	27	0.276 49	1.329 36	0.999 11	1.028 36

In the second method, due to Adler *et al* (1981), one studies Dlog Padé approximants to the function $G(q) = \beta P(q) + (q_c - q)\,\mathrm{d}P(q)/\mathrm{d}q$. The logarithmic derivative to $G(q)$

has a pole at q_c with residue $\beta + \Delta$. We evaluate the Dlog Padé approximants for a range of guesses for q_c and β. For each such guess we thus find an estimate for Δ; for the correct value of q_c and β one would expect to see a convergence region in (q_c, β, Δ)-space. In practice we always froze either q_c or β and examined Δ as a function of the other parameter. Figure 5 shows, respectively, Δ as a function of q_c with $\beta = 0.276\,43$ and Δ as a function of β with $q_c = 0.355\,299\,4$. This analysis clearly supports $\Delta \simeq 1$, and thus that there is no sign of any non-analytic corrections to scaling.

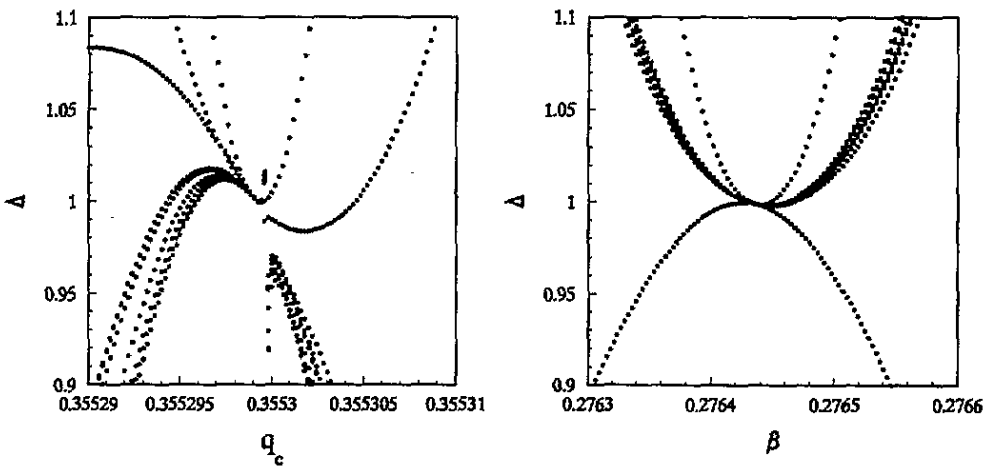

Figure 5. The confluent exponent Δ as a function of, respectively, the parameter q_c (with $\beta = 0.276\,43$) and the parameter β (with $q_c = 0.355\,299\,4$) using the method of Adler *et al* (1981).

For the square site series the results from the Baker–Hunter transformation is less convincing, as there is no value of q_c at which the various aprroximants cross. If we look closely at the approximants evaluated at $q_c = 0.294\,515$ we find, generally speaking, that only the $[N-1, N]$ approximants yield estimates of β close to the expected value with corresponding estimates for Δ consistent with an analytic correction. The method of Adler *et al* (1981) confirms that $\Delta \simeq 1$.

In the honeycomb bond case several of the approximants to the Baker–Hunter transformed series have a crossing for $q_c = 0.177\,144(1)$, $\beta = 0.2767(1)$ and $\Delta = 0.89(2)$, though it should be noted that the scatter is quite large. When we analyse the series using the second method we find that, for q_c and β close to the central values from the Dlog Padé analysis, a value of 1 for Δ is fully compatible with the results.

5. Conclusion

In this paper we have presented extended series for the percolation probability for site and bond percolation on the square and honeycomb lattices. The analysis of the series leads to improved estimates for the percolation threshold (particularly for the honeycomb bond

problem) and the order-parameter exponent β. To summarize we estimate that

$q_c = 0.355\,299\,4(10)$	$\beta = 0.276\,43(10)$	$A = 1.3292(5)$ square bond problem
$q_c = 0.294\,515(5)$	$\beta = 0.2763(3)$	$A = 1.425(1)$ square site problem
$q_c = 0.177\,143(2)$	$\beta = 0.2763(2)$	$A = 1.106(1)$ honeycomb bond problem
$q_c = 0.160\,067(5)$	$\beta = 0.2763(4)$	$A = 1.167(1)$ honeycomb site problem.

The estimates for $q_c = 1 - p_c$ for the square bond and site problem are in excellent agreement with those obtained by Essam *et al* (1986, 1988), $q_c = 0.355\,303(6)$ and $q_c = 0.294\,51(1)$, respectively. The estimates for β clearly show, as one would expect, that all the models studied in this paper belong to the same universality class. The value of β does not suggest any simple fraction. Indeed, around the central value for β (square bond), we find only four fractions with denominators less than 1500. They are: $\frac{34}{123} = 0.276\,422\ldots$, $\frac{387}{1400} = 0.276\,429\ldots$, $\frac{217}{785} = 0.276\,433\ldots$ and $\frac{183}{662} = 0.276\,435\ldots$. None of these are remotely compelling, and leave open the question as to why this apparently simple problem has such an ugly exponent. This does seem to be a frequent characteristic of directed problems, as evidenced by the recent study of the longitudinal size exponent of square lattice directed animals (Conway and Guttmann 1994) in which it was found that $\nu_{\parallel} = 0.817\,22(5)$, a result which suggests no simple rational fraction. Finally, we note that none of the series show any evidence of non-analytic confluent correction terms. This provides a hint that the model might be exactly solvable.

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References

Adler J, Moshe M and Privman V 1981 *J. Phys. A: Math. Gen.* **17** 2233
Baker G A and Hunter D L 1973 *Phys. Rev.* B **7** 3377
Baxter R J and Guttmann A J 1988 *J. Phys. A: Math. Gen.* **21** 3193
Bidaux R and Forgacs G 1984 *J. Phys. A: Math. Gen.* **17** 1853
Blease J 1977 *J. Phys. C: Solid State Phys.* **10** 917
Bousquet-Mélou M 1995 Percolation models and animals *Preprint*
Broadbent S R and Hammersley J M 1957 *Proc. Camb. Phil. Soc.* **53** 629
Cardy J L and Sugar R L 1980 *J. Phys. A: Math. Gen.* **13** L423
Conway A R and Guttmann A G 1994 *J. Phys. A: Math. Gen.* **27** 7007
De'Bell K and Essam J W 1983a *J. Phys. A: Math. Gen.* **16** 3145
De'Bell K and Essam J W 1983b *J. Phys. A: Math. Gen.* **16** 3553
Delest, M-P and Fédou, J M 1989 Exact formulas for fully diagonal compact animals *Internal Report* LaBRI 89-06, Univ. Bordeaux I, France
Dhar D, Phani M K and Barma M 1982 *J. Phys. A: Math. Gen.* **15** L279
Essam J W and De'Bell K 1982 *J. Phys. A: Math. Gen.* **15** L601
Essam J W, De'Bell K, Adler J and Bhatti F M 1986 *Phys. Rev.* B **33** 1982
Essam J W, Guttmann A J and De'Bell K 1988 *J. Phys. A: Math. Gen.* **21** 3815
Grassberger P and de la Torre A 1979 *Ann. Phys., NY* **122** 373
Grassberger P and Sundermeyer K 1978 *Phys. Lett.* **77B** 220
Guttmann A J 1989 Asymptotic analysis of power-series expansions *Phase Transitions and Critical Phenomena* vol 13 ed C Domb and J L Lebowitz (New York: Academic)
Harris T E 1974 *Ann. Prob.* **2** 969
Jensen I 1994 *J. Phys. A: Math. Gen.* **27** L61
Kertész J and Viscek T 1980 *J. Phys. C: Solid State Phys.* **13** L343
Kinzel W 1985 *Z. Phys.* B **58** 229

Knuth D E 1969 *Seminumerical Algorithms (The Art of Computer Programming 2)* (Reading, MA: Addison-Wesley)
Köhler J and ben-Avraham D 1991 *J. Phys. A: Math. Gen.* **24** L621
Liggett T M 1985 *Interacting Particle Systems* (New York: Springer)
Liu A J and Fisher M E 1989 *Physica* **156A** 35
Obukhov S P 1990 *Phys. Rev. Lett.* **65** 1395
Onody R N 1990 *J. Phys. A: Math. Gen.* **23** L335
Onody R N and Neves U P C 1992 *J. Phys. A: Math. Gen.* **25** 6609
Paczuski M, Maslov S and Bak P 1994 *Europhys. Lett.* **27** 97
Redner S and Brown A C 1981 *J. Phys. A: Math. Gen.* **14** L285
Schlögl F 1972 *Z. Phys.* **252** 147
Schulman L S and Seiden P E 1982 *J. Stat. Phys.* **27** 83
Viennot X G 1994 Private communication
Zhuo J, Redner S and Park H 1993 *J. Phys. A: Math. Gen.* **26** 4197
Ziff R M, Gulari E and Barshad Y 1986 *Phys. Rev. Lett.* **56** 2553